Noise reduction in signal processing using binary couplings

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We report a simple extension of a model for noise reduction in signal processing, already introduced by Mourik *et al.*, in the presence of binary coupling vectors, which turns to be more useful for practical and engineering implementations. We also compute annealed approximation which gives an upper bound of the correct critical number of noise-sources. Finally, we also find that the full RSB Parisi solution just above the AT line is an universal function for any symmetric distribution of the coupling vectors.

PACS numbers: 64.60.Cn, 89.70.+c,87.10.+e, 02.50.-r

I. THE MODEL

In a recent paper [1], Mourik *et al.* investigated the statistical properties for the problem of noise reduction (NR) in signal processing. For this model one considers N-detectors receiving a signal mixed with noise from p sources. The total input on a source from the detector j reads

$$x_j = a_j S + \sum_{\mu} \xi_j^{\mu} n_{\mu} \tag{1}$$

where S and n_μ are respectively the amplitudes of signal and of the noise from μ-th source, ξ_j's are random gaussianly distributed variables, and a_j's fixed numbers. The goal is to find a linear combination of the inputs in order to reduce the noise and at the same time have a clear signal. By using a N-dimensional weight vector as a "filter", the sum of observable inputs x_j weighted with such a vector, reads

$$h \equiv \frac{1}{\sqrt{N}} \sum_j J_j x_j = \frac{S}{\sqrt{N}} \sum_j J_j a_j + \sum_{\mu} \sum_j \frac{J_j \xi_j^{\mu}}{\sqrt{N}}. \tag{2}$$

Now, in order to reduce the noise from the local field $h^{\mu} = \sum_j J_j \xi_j^{\mu}/\sqrt{N}$ and have the signal part $S \sum_j J_j a_j/\sqrt{N}$ constant $\sim\mathcal{O}(1)$, we use the strategy which directly minimize the quantity $|h^{\mu}|$. If the number of the noise sources p is small, it is straightforward to find the optimal solution which minimizes the "cost" $|h^{\mu}|$. However, as the number of the noise sources increases, in general, it becomes hard to solve such optimization problem. In this case, it is natural to wonder how many configurations $\boldsymbol{J} = \pm 1$ do exist when the phase space is restricted to $|h^{\mu}| < k$, where k is the noise tolerance. For each configuration of the noise, the number of the solutions $\boldsymbol{J}$ which satisfy the previous condition can be expressed as

$$\mathcal{N}(\boldsymbol{J}) = \mathrm{Tr}_{\{\boldsymbol{J}\}} \prod_{\mu=1}^{p} \Theta(k^2 - (h^{\mu})^2) \tag{3}$$

and the "entropy" of the solution space is given by the its logarithm $\mathcal{S}(\boldsymbol{J}) = \log\mathcal{N}(\boldsymbol{J})$.

In order to investigate the typical properties of the entropy, one usually averages the above quantity with respect to the gaussian noise source $\boldsymbol{\xi}$, namely

$$< \mathcal{S}(\boldsymbol{J}) >_{\boldsymbol{\xi}} = < \log\mathcal{N}(\boldsymbol{J}) >_{\boldsymbol{\xi}} . \tag{4}$$

The previous quenched average can be performed by means of the usual replica method [3], thus giving

$$< \mathcal{N}^n(\boldsymbol{J}) >_{\boldsymbol{\xi}} = < \mathrm{Tr}_{\{\boldsymbol{J}\}} \prod_{\mu=1}^{p} \prod_{a=1}^{n} \Theta(k^2 - (h_a^{\mu})^2) >_{\boldsymbol{\xi}} . \tag{5}$$

Therefore, after performing the average with respect to the gaussian randomness and introducing auxiliary variables of integrations, the free energy simply reads

$$
\begin{aligned}
&\exp[f] \sim \int \prod_{\alpha<\beta} dq_{\alpha\beta} d\tilde{q}_{\alpha\beta} \\
&\times \exp\left[N\left(\alpha\mathcal{G}_0(\{q\}) + \mathcal{G}_1(\{\tilde{q}\}) + \frac{1}{2}\sum_{\alpha\neq\beta} q_{\alpha\beta}\tilde{q}_{\alpha\beta}\right)\right]
\end{aligned} \tag{6}
$$

where

$$
\begin{aligned}
&\exp\left[\mathcal{G}_0(\{q\})\right] \\
&= \int_{-\kappa}^{\kappa} [d\lambda] \int_{-\infty}^{\infty} [dh] \exp\left[-\frac{1}{2}\sum_{\alpha} h_\alpha^2 - \frac{1}{2}\sum_{\alpha\neq\beta} h_\alpha q_{\alpha\beta} h_\beta\right]
\end{aligned} \tag{7}
$$

and

$$
\exp\left[\mathcal{G}_1(\{\tilde{q}\})\right] = \mathrm{Tr}_{\{\boldsymbol{J}\}} \exp\left[\frac{1}{2}\sum_{\alpha\neq\beta} J_\alpha \tilde{q}_{\alpha\beta} J_\beta\right]. \tag{8}
$$

As usual, the previous quantities should be evaluated in the limit $N\to\infty$, by making an ansatz for the matrices $q_{\alpha\beta}$ and $q_{\tilde{\alpha\beta}}$. In the next section, we consider the replica symmetric solution.

II. RS SOLUTION

Within the replica symmetric ansatz

$$
q_{\alpha\beta} \equiv \frac{1}{N}\sum_j J_j^\alpha J_j^\beta = q \qquad \tilde{q}_{\alpha\beta} = \tilde{q} \quad \forall \alpha,\beta \tag{9}
$$

it is straightforward to check that the entropy per system size N reads, in the limit $n\to 0$

$$
\begin{aligned}
&\frac{\log < \mathcal{N}(\boldsymbol{J}) >_{\boldsymbol{\xi}}}{Nn} \equiv \mathcal{S}(q,\tilde{q}) \\
&= \mathrm{extr}_{q,\tilde{q}}\Big[\alpha \int Dy \log\left[H\left(\frac{-k+\sqrt{q}y}{\sqrt{1-q}}\right) - H\left(\frac{k+\sqrt{q}y}{\sqrt{1-q}}\right)\right] \\
&- \frac{\tilde{q}}{2}(1-q) + \int Dt \log 2\cosh(\sqrt{\tilde{q}}t)\Big].
\end{aligned} \tag{10}
$$

The stationary points of the entropy with respect to q and $\tilde{q}$

$$
\frac{\partial\mathcal{S}(q,\tilde{q})}{\partial q} = \frac{\partial\mathcal{S}(q,\tilde{q})}{\partial\tilde{q}} = 0. \tag{11}
$$

give the optimal number of the noise sources as a function of the noise tolerance k. Moreover, in order to obtain the critical number of the noise source α_{c}, beyond which the solution space shrinks down to zero, we have to estimate the expression (11) in the limit $q\to 1$, *viz.*

$$
\alpha_c^{(0)}(k) = \frac{2}{\pi}\left[2(1+k^2)H(k) - \sqrt{\frac{2}{\pi}}k\exp(-\frac{k^2}{2})\right]^{-1} \tag{12}
$$

We thus get the same result as [1] except for a factor $2/\pi$, just as in the case of the random pattern storage problems [2,4]. Therefore, one obtains $\alpha_{\mathrm{c}}^{(0)}(k) \simeq \sqrt{2/\pi}k^3 \mathrm{e}^{k^2/2}/2$ (large k limit) and $\alpha_c^{(0)}(k) \sim 2\left(1+4k/\sqrt{2\pi}\right)/\pi$ (small k limit).

At fixed k, it is important to compare the value of the critical capacity with the AT line, beyond which the saddle point solution turns to be unstable against transverse fluctuations. Because of the inversion symmetry of the constraints, the saddle point equations (11) are satisfied with the solution $q=0$, independent of α and k. The stability

of such solution is determined by studying the eigenvalues of the Hessian around the saddle point solution. Therefore, the AT line is found to be

$$\alpha_{\rm AT}(k) = \frac{\pi}{2}\frac{[1-2H(k)]^2}{k^2{\rm e}^{-\frac{k^2}{2}}} \tag{13}$$

as in the continuous case. So, qualitatively we get the same results as [1] where a discontinuous phase transition seems to appear for large k.

In order to give an upper bound for $\alpha_{\rm c}(k)$, we also investigate the optimal number of the noise source using annealed approximation [4]. So we replace the quenched average by the following annealed average

$$< \mathcal{S}(\boldsymbol{J}) >_{\boldsymbol{\xi}} = \log < \mathcal{N}(\boldsymbol{J}) >_{\boldsymbol{\xi}} . \tag{14}$$

Then we introduce the "magnetization" $\mathcal{M} = \sum_j J_j/\sqrt{N}$ and rewrite the number of the configurations $\mathcal{N}$ as

$$\mathcal{N}(\boldsymbol{J}) = {\rm Tr}_{\{\boldsymbol{J}\}}\prod_{\mu=1}^{p}\Theta(k^2-(h^\mu)^2)\delta(\sqrt{N}\mathcal{M}-\sum_j J_j). \tag{15}$$

It should be noticed that in the quenched calculation, the contribution of the magnetization $\mathcal{M}$ in the entropy becomes $\mathcal{O}(\sqrt{N})$ and vanishes in the thermodynamical limit. In the annealed calculation, the average with respect to $\boldsymbol{\xi}$ is a simple gaussian integral and we obtain the entropy per system size as

$$\mathcal{S}^{\rm anneal}(\tilde{\mathcal{M}}) = \alpha\log\left[1-H(k)\right] + \log\left(2{\rm cosh}(\tilde{\mathcal{M}})\right). \tag{16}$$

The saddle point equation with respect to $\tilde{\mathcal{M}}$, that is, $\partial\mathcal{S}/\partial\tilde{\mathcal{M}} = 0$ leads to $\tilde{\mathcal{M}} = 0$, and finally we obtain

$$\mathcal{S}^{\rm anneal} = \alpha\log\left[1-2H(k)\right] + \log 2. \tag{17}$$

As it is well known, due to the concavity of the logarithm, the annealed free energy is a lower bound for the correct free energy. From this argument and the fact that the quenched entropy cannot be negative, we obtain a lower bound on the zero temperature entropy $\mathcal{S}^{\rm anneal}$. From (17), we get $\alpha_{\rm c}^{\rm anneal}(k)$, which reads

$$\alpha_{\rm c}(k) \leq -\frac{\log 2}{\log[1-2H(k)]} = \alpha_{\rm c}^{\rm anneal}. \tag{18}$$

According to this rigorous condition, RS result $\alpha_{\rm c}^{(0)}$ turns out to be larger than $\alpha_{\rm c}^{\rm anneal}$. Therefore, we conclude that RS solution is not enough to estimate $\alpha_{\rm c}$ correctly and more steps of RSB are needed. We plotted $\alpha_{\rm c}^{\rm anneal}$ in FIG. 1. For large k we get $\alpha_{\rm c}^{\rm anneal}(k) \simeq \left(\sqrt{2/\pi}\log 2\right)k{\rm e}^{k^2/2}$, and for small k $\alpha_{\rm c}^{\rm anneal}(k) \simeq -\log 2/\log\left(\sqrt{2/\pi}k\right)$.

In order to estimate the critical number of the noise source, $\alpha_{\rm c}$, we may use the another criterion, that is, $\alpha_{\rm s}$ at which the zero temperature entropy of the RS saddle point becomes negative. By numerical calculations, we checked that the entropy S of the RS saddle point becomes negative at $q = \tilde{q} = 0$ for all k. This leads to

$$\alpha_{\rm s} = \alpha_{\rm c}^{\rm anneal} \tag{19}$$

and this RS solution is locally stable. Therefore, we conclude that $\alpha_{\rm s}$ gives one of the good approximations for the critical number of the noise source.

III. PARISI SOLUTION

As in [1] the finite step RSB solution is found to be unstable against traverse fluctuations, signaling the necessity of a full RSB solution. Analytical results can be obtained only close to the AT line, where the order parameters are small ($q_{\alpha\beta} \sim (\alpha-\alpha_{AT})/\alpha_{AT}$), and then the free energy, or equivalently the saddle point equations, can be expanded in terms of $q_{\alpha\beta}$ and $\tilde{q}_{\alpha\beta}$. The expressions involved in here are quite similar to the SK model, where for small value of the order parameter, the free energy has, besides a quadratic term, a cubic term ${\rm Tr}q^3$ and a quartic term $\sum_{\alpha\beta}q_{\alpha\beta}^4$, whose presence allows a full-fledged replica symmetry breaking solution [5]. It is straightforward to see that the saddle point equations, obtained by varying (6) with respect to $q_{\alpha\beta}$ and $\tilde{q}_{\alpha\beta}$, coincide with the equation for the continuum

$\boldsymbol{J}$ case, thus giving the same Parisi solution, *i.e.* linear up to a breakpoint and then constant$q = q_{EA}$, for any value of k as long as $k^2 q_{EA} \ll 1$.

Finally, it is worth noticing that the saddle point equation with respect to $\tilde{q}_{\alpha\beta}$, involving the distribution of the couplings $\boldsymbol{J}$, gives, in the proximity of the AT line, $q_{\alpha\beta} \sim \tilde{q}_{\alpha\beta}$ for any kind of symmetric distribution of the coupling vectors $\boldsymbol{J}$. Therefore we conclude that the Parisi solution just above the AT line is an universal function, whereas sizable differences may arise far away from the AT line.

IV. ACKNOWLEDGMENT

The authors acknowledge stimulating discussion with H. Nishimori, J. van Mourik and K. Y. M. Wong.
D.M.C.'s research was supported by the JSPS under Grant No. P96215.

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FIG. 1. RS solution, annealed approximation (AA), AT line and RS solution by Mourik *et al.*

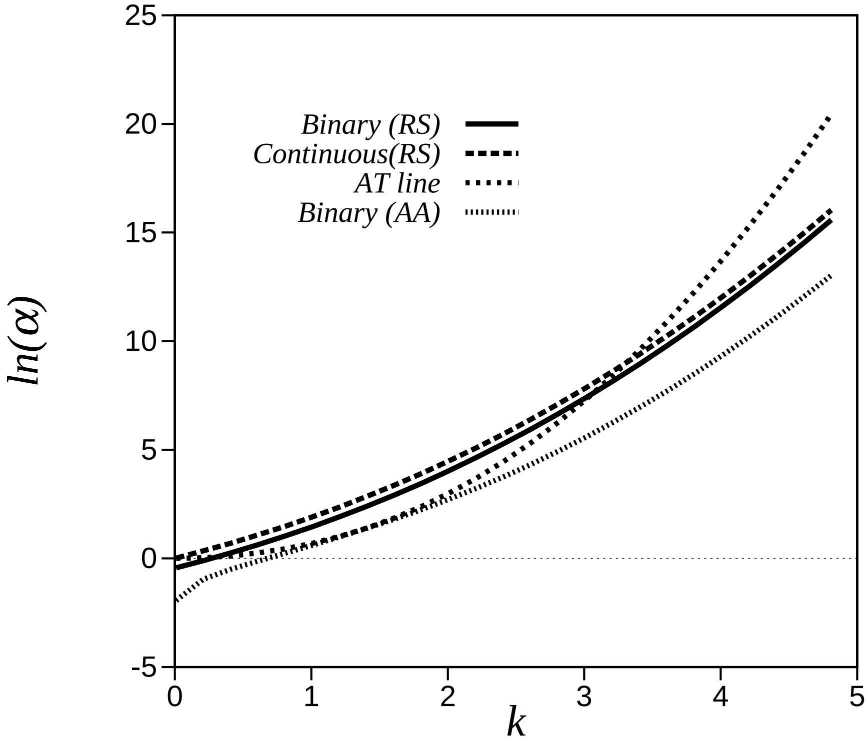

FIG. 1

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